Indian Statistical Institute, Bangalore Centre

B.Math (Hons) II Year, Second Semester

Semestral Examination

Optimization

Time: 3 Hours May 10, 2013 Instructor: Pl.Muthuramalingam

The paper carries 50 marks. You need to answer for 45 marks.

1. (a) Let U be a convex set in R^n, $\{g_\alpha : \alpha \in I\}$ a finite or infinite family of convex functions : $U \longrightarrow R$. Assume that $h(u) = \sup_\alpha g_\alpha(u)$ is finite for each u in U. Show that h is convex. [2]

   (b) Let $f : R^2 \longrightarrow R$ be given by $f(x_1, x_2) = (x_1^2 + x_2^2)^{1/2}$. Show that f is a convex function. [2]

2. State and prove the first order condition of Lagrange method of multipliers. [7]

3. (The aim is to prove strong duality theorem.) Let $A : R^n_{col} \longrightarrow R^{m_0}_{col}$ be a linear onto map. Let $A = [\underset{\sim}{a}^1, \underset{\sim}{a}^2, \cdots, \underset{\sim}{a}^n]$ where $\underset{\sim}{a}^i$ is a column vector. Let $B = [\underset{\sim}{a}^1, \underset{\sim}{a}^2, \cdots, \underset{\sim}{a}^{m_0}]$ be a basis.

   For
$$\ell \text{ in } \{m_0 + 1, m_0 + 2, \cdots, n\}, \text{ let } \underset{\sim}{a}^\ell = \sum_{j=1}^{m_0} r_j^\ell \, \underset{\sim}{a}^j$$

   For
$$1 \le j \le m_0 \le m_0 + 1 \le l \le n, \text{ let } B(\ell, j) = BU\{\underset{\sim}{a}^\ell\} \backslash \{\underset{\sim}{a}^j\}$$

   (a) If $r_j^\ell \neq 0$ show that $B(l, j)$ is a basis. [The converse is also true, but do not prove]. [2]

   (b) Let $\underset{\sim}{x}$ be B basic, and $\underset{\sim}{y}$ be $B(l, j)$ basic solution for $A\underset{\sim}{u} = \underset{\sim}{b}$ where $\underset{\sim}{b}$ is a given fixed vector. Show that $y_\ell = x_j / r_j^\ell$, $y_i = x_i - y_\ell \, r_i^\ell$ for $i \neq \ell$ [1]

   (c) Let $\underset{\sim}{x}, \underset{\sim}{y}$ be as above. For any cost vector $\underset{\sim}{c}$ given by $\underset{\sim}{c}^t = (c_1, c_2, \cdots, c_{m_0}, c_{m_{0+1}}, \cdots, c_n)$, show that $\underset{\sim}{c}^t y - \underset{\sim}{c}^t \underset{\sim}{x} = y_\ell \left[c_\ell - \sum_{p \neq \ell} c_p \, r_p^\ell \right]$. [1]

   (d) Assume that the B basic feasible solution $\underset{\sim}{x}$ satisfies $\underset{\sim}{c}^t \underset{\sim}{x} = \inf\{\underset{\sim}{c}^t u : A\underset{\sim}{u} = \underset{\sim}{b}, \underset{\sim}{u} \geq \underset{\sim}{0}\}$.

   Assume that for each ℓ_0 in $\{m_0 + 1, m_0 + 2, \cdots, n\}$, there exists j_0 in $\{1, 2, \cdots m_0\}$ such that $B(\ell_0, j_0)$ is a base and has a basic feasible solution. Then show that
$$c_\ell - \sum_{p \neq \ell} c_p \, r_p^\ell \geq 0 \qquad (I_1)$$

   for each ℓ in $\{m_0 + 1, m_0 + 2, \cdots n\}$. [1]

(e) Define $\underset{\sim}{c}^*$, truncation of $\underset{\sim}{c}$, by $\underset{\sim}{c}^{*^t} = (c_1, c_2, \cdots c_{m_0})$. If I_1 holds, show that $\underset{\sim}{c} - A^t(B^{-1})^t \underset{\sim}{c}^* \geq 0$ (I_2). [2]

(f) Let $\underset{\sim}{x}$ be as in (d). If (I_2) holds show that strong duality theorem holds. [2]

4. (a) For $B : R^n_{col} \longrightarrow R^{m_0}_{col}$ linear map, show that $(Bu)'Bu \leq \|B\|^2\, u'u$ for all $u \in R^n_{col}$ where $\|B\|^2 = \sum\limits_{i,j} b^2_{ij}$, $B = ((b_{ij}))$. [1]

   (b) Let $A_0, A_1, A_2, \cdots$ be sequence of real symmetric $n \times n$ matrices. Assume that (ij) the entry of A_k converges to (ij) entry of A_0 as $k \longrightarrow \infty$. Let L be any linear subspace of R^n. Assume that $\inf\{u'A_0u : u \in L, u'u = 1\} = p_0 > o$. Show that there exists k_0 such that for all $k \geq k_0$, we have $u'A_ku \geq \frac{p_0}{2}\, u'u$ for all $u \in L$. [2]

5. Let $A : R^n_{col} \longrightarrow R^{m_0} col$ be linear onto. Let $\underset{\sim}{z} = (z_1, z_2, \cdots z_n)$, $z_i > 0$ for each i with $\sum z_i = 1$. Let $A\underset{\sim}{z} = 0$. Let $D =$ diag $(z_1, z_2, \cdots, z_n)$. Define $B : R^n_{col} \longrightarrow R^{m_0+1}_{col}$ by $B = \begin{bmatrix} AD \\ (1, 1, \cdots, 1) \end{bmatrix}$

   (a) Show that BB^t is invertible. [5]

   (b) For $Q = B^t(BB^t)^{-1}B$, prove $Q = Q^2 = Q^t$. [2]

   (c) If $P = 1 - Q$, then Range $P =$ null B. [1]

6. Let $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ and $g(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$. Find $\inf\{f(\underset{\sim}{x}) : g(\underset{\sim}{x}) = 1\}$ using the first order and second order conditions of Lagrange. [5]

7. Let $f(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$, $g_1(x_1, x_2, x_3) = 4 - (x_1^2 + x_2^2 + x_3^2)$, and $g_2(x_1, x_2, x_3) = 1 - (x_1 + x_2 + x_3)$. Using Kuhn-Tucker-Karush theorem, study the existence of local maxima or minima for the function f on the given sets and then find them.

   (a) $\{g_2 = 0\} \cap \{g_1 > 0\}$. [6]

   (b) $\{g_2 > 0\} \cap \{g_1 = 0\}$. [4]

8. Write an essay on <u>One</u> of the topics : [3]

   (a) Diet problem

   (b) Transportation problem

   (c) Matching problem

   (d) Any topic interesting in this course, other than found in questions 1 to 7

Duality Table

	Primal	Dual Table
	$A, \mathbf{x}, \mathbf{b}, \mathbf{c}$	$A^t, \mathbf{y}^t, \mathbf{c}^t, \mathbf{b}^t$
$i \varepsilon I_1,$	$\sum_j a_{ij} x_j = b_i$	$y_i real, y_i \gtreqless 0$
$i \varepsilon I_2,$	$\sum_j a_{ij} x_j \geq b_i$	$y_i \geq 0$
$i \varepsilon I_3,$	$\sum_j a_{ij} x_j \leq b_i$	$y_i \leq 0$
$j \varepsilon J_1,$	$x_j real, x_j \gtreqless 0$	$\sum_i y_i a_{ij} = c_j$
$j \varepsilon J_2,$	$x_j \geq 0$	$\sum_i y_i a_{ij} \leq c_j$
$j \varepsilon J_3,$	$x_j \leq 0$	$\sum y_i a_{ij} \geq c_j$
	$min \sum_j c_j x_j$	$max \sum_i y_i b_i$